Comment on “Absence of trapped surfaces and singularities in cylindrical collapse”

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Recently, the gravitational collapse of an infinite cylindrical thin shell of matter in an otherwise empty spacetime with two hypersurface orthogonal Killing vectors was studied by Gonçalves [Phys. Rev. **D65**, 084045 (2002).]. By using three “alternative” criteria for trapped surfaces, the author claimed to have shown that *they can never form either outside or on the shell, regardingless of the matter content for the shell, except at asymptotical future null infinite*.

Following Penrose’s original idea, we first define trapped surfaces in cylindrical spacetimes in terms of the expansions of null directions orthogonal to the surfaces, and then show that the first criterion used by Gonçalves is incorrect. We also show that his analysis of non-existence of trapped surfaces in vacuum is incomplete. To confirm our claim, we present an example that is a solution to the vacuum Einstein field equations and satisfies all the regular conditions imposed by Gonçalves. After extending the solution to the whole spacetime, we show explicitly that trapped surfaces exist in the extended region.

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I. INTRODUCTION

The gravitational collapse of an infinite cylindrical thin shell of generic matter in an otherwise empty spacetime with two hypersurface orthogonal Killing vectors was recently studied by Gonçalves [1]. After re-deriving the junction conditions across an infinitely thin shell with the spacetime being vacuum both inside and outside the shell, the author studied the formation of trapped surfaces by three “alternative” criteria, the so-called proper circumference radius criterion, the specific area radius criterion, and the criterion of geodesic null congruences. The author claimed that *trapped surfaces can never form either out of the shell or on the shell, except at asymptotic future null infinity.*

In this Comment, following Penrose’s original idea [2], we shall first define trapped surfaces in cylindrically symmetric spacetimes in terms of the expansions of the two null directions orthogonal to the surfaces. Then we shall show that the first criterion used by Gonçalves for the formation of trapped surfaces is incorrect. We shall also show that Gonçalves’ analysis of non-existence of trapped surfaces in vacuum by the second and third criteria is incomplete. The reason is simply because the Einstein-Rosen coordinates used by Gonçalves usually cover only part of a spacetime, quite similar to the Schwarzschild coordinates in the spherically symmetric spacetimes. To show this explicitly, we present an example that is a solution to the vacuum Einstein field equations and satisfies all the conditions that Gonçalves imposed for a spacetime to be cylindrical. In the Einstein-Rosen coordinates, trapped surfaces indeed do not exist. But, after extending the solution to the whole spacetime, we do find trapped surfaces in the extended region.

II. TRAPPED SPATIAL TWO-SURFACES IN CYLINDRICAL SPACETIMES

The general metric for cylindrical spacetimes with two hypersurface orthogonal Killing vectors takes the form [3],

$$ds^2 = e^{2(\gamma-\psi)}\left(dr^2 - dt^2\right) + e^{2\psi}dz^2 + \alpha^2 e^{-2\psi}d\varphi^2, \tag{1}$$

where γ, ψ and α are functions of t and r only, and $x^\mu = \{t, r, z, \varphi\}$ are the usual cylindrical coordinates, and the hypersurfaces $\varphi = 0, 2\pi$ are identified. To have cylindrical symmetry, some conditions needed to be imposed. In general this is not trivial [4]. Gonçalves defined cylindrical spacetimes by the existence of the two commuting

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spacelike Killing vector fields, $\xi_{(2)} \equiv \partial_z$ and $\xi_{(3)} \equiv \partial_\varphi$. At the symmetry axis $r = 0$, the local-flatness condition was also imposed,

$$\frac{X_{,\alpha}X_{,\beta}g^{\alpha\beta}}{4X} \to 1, \tag{2}$$

as $r \to 0^+$, where $(\ )_{,\alpha} \equiv \partial(\ )/\partial x^\alpha$, $X \equiv \xi^\mu_{(3)}\xi^\nu_{(3)}g_{\mu\nu}$, and the radial coordinate is chosen such that the symmetry axis is at $r = 0$.

The concept of *trapped surfaces* was originally from Penrose [2], who defined it as a *compact* spatial two-surface S on which $\theta_+\theta_-|_S > 0$, where $\theta_\pm$ denote the expansions in the future-pointing null directions orthogonal to S, and the spacetime is assumed to be time-orientable, so that "future" and "past" can be assigned consistently. One may then define a past trapped surface by $\theta_\pm|_S > 0$, and a future trapped surface by $\theta_\pm|_S < 0$.

Recently, Hayward generalized the above definition to the cylindrical spacetimes where the two-surface S is *not compact* but an infinitely long two-cylinder of constant t and r, and call it trapped, marginal or untrapped, according to where $\alpha_{,\mu}$ is timelike, null or spacelike [5]. This definition is the second criterion used in [1], and was referred to as the specific area radius criterion.

In the vacuum case, one of the Einstein field equations can be written as [3]

$$\alpha_{,tt} - \alpha_{,rr} = 0, \tag{3}$$

which has the general solution

$$\alpha(t,r) = F(t+r) - G(t-r), \tag{4}$$

where $F(t+r)$ and $G(t-r)$ are arbitrary functions of their indicated arguments. In the Einstein-Rosen coordinates, the functions F and G are chosen as $F(t+r) = (t+r)/2$ and $G(t-r) = (t-r)/2$, so that

$$\alpha = r. \tag{5}$$

Working with the Einstein-Rosen gauge, Gonçalves calculated the expansions in the future-pointing null directions orthogonal to a cylinder of constant t and r, and found that

$$\theta^{Gon.}_\pm = (\partial_t \mp \partial_r)\left(re^{-\psi}\right) = -re^{-\psi}\left((\partial_t \mp \partial_r)\,\psi \pm \frac{1}{r}\right). \tag{6}$$

These expressions are wrong and can be seen clearly from Eq.(B7) given in [6], which in terms of the present notation takes the form

$$\begin{aligned} \theta_+ &\equiv g^{\alpha\beta}l_{\alpha;\beta} = e^{-2\sigma}\frac{\alpha_{,v}}{\alpha} = e^{2(\psi-\gamma)}\frac{\alpha_{,t}+\alpha_{,r}}{2b'(u)\alpha}, \\ \theta_- &\equiv g^{\alpha\beta}n_{\alpha;\beta} = e^{-2\sigma}\frac{\alpha_{,u}}{\alpha} = e^{2(\psi-\gamma)}\frac{\alpha_{,t}-\alpha_{,r}}{2a'(v)\alpha}, \end{aligned} \tag{7}$$

where

$$t \equiv a(v) + b(u), \quad r \equiv a(v) - b(u), \tag{8}$$

with $a(v)$ and $b(u)$ being arbitrary functions of their indicated arguments, subject to $a'(v)b'(u) > 0$. A prime denotes the ordinary differentiation. The two null vectors l_λ and n_λ are future-directed, orthogonal to the two-cylinder of constant t and r, and given by

$$\begin{aligned} l_\lambda &\equiv \delta^u_\lambda = \frac{1}{2b'(u)}\left(\delta^t_\lambda - \delta^r_\lambda\right), \\ n_\lambda &\equiv \delta^v_\lambda = \frac{1}{2a'(v)}\left(\delta^t_\lambda + \delta^r_\lambda\right). \end{aligned} \tag{9}$$

For the details, we refer readers to [6]. It should be noted that the two null vectors are uniquely defined only up to a factor [7]. In fact, $\bar{l}_\mu = f(u)\delta^u_\mu$ and $\bar{n}_\mu = g(v)\delta^v_\mu$ represent another set of null vectors that also define affinely parameterized null geodesics, and the corresponding expansions are given by $\bar{\theta}_+ = f(u)\theta_+$ and $\bar{\theta}_- = g(v)\theta_-$. However, since along each curve $u = Const.$ ($v = Const.$) $f(u)$ ($g(v)$) is constant, this does not affect the definition of trapped

surfaces in terms of the expansions (See [7] and the discussions given below). Thus, without loss of generality, in the following we consider only the expressions given by Eq.(7).

Once we have $\theta_{\pm}$, following Penrose and Hayward we can define that *a two-cylinder, S, of constant t and r is trapped, marginally trapped, or untrapped, according to whether $\theta_+\theta_- > 0$, $\theta_+\theta_- = 0$, or $\theta_+\theta_- < 0$. An apparent horizon, or trapping horizon in Hayward's terminology [8], is defined as a hypersurface foliated by marginally trapped surfaces.* It is said *outer, degenerate, or inner*, according to whether $\mathcal{L}_n\theta_+|_\Sigma < 0$, $\mathcal{L}_n\theta_+|_\Sigma = 0$, or $\mathcal{L}_n\theta_+|_\Sigma > 0$, where $\mathcal{L}_n$ denotes the Lie derivative along the normal direction n_μ. In addition, if $\theta_-|_\Sigma < 0$ then the apparent horizon is said *future*, and if $\theta_-|_\Sigma > 0$ it is said *past* [8].

On the other hand, from Eq.(7) we find that

$$\alpha_{,\nu}\alpha^{,\nu} = -2\alpha^2 e^{2\sigma}\theta_+\theta_-, \tag{10}$$

where $\sigma \equiv \gamma - \psi + \frac{1}{2}\ln(2a'(v)b'(u))$ and is finite on non-singular surfaces [1]. Then, from Eq.(10) we can see that the above definition given in terms of the expansions, θ_+ and θ_-, is consistent with that given by Hayward who defined it according to the nature of the vector $\alpha_{,\mu}$ [5]. This definition is the second criterion used in [1].

Note that Eq.(7) is valid for any spacetime where the metric is given by Eq.(1). This, in particular, includes the cases where the spacetimes are not vacuum. Thus, to compare Eq.(6) with Eq.(7) we need first to restrict ourselve to the vacumm case and then choose the Einstein-Rosen gauge (5), for which Eq.(7) reduces to

$$\begin{aligned} \theta_+ &= f(u)\frac{1}{r}e^{2(\psi-\gamma)}, \\ \theta_- &= -g(v)\frac{1}{r}e^{2(\psi-\gamma)}, \quad (\alpha = r), \end{aligned} \tag{11}$$

where $f(u) = 1/(2b'(u))$ and $g(v) = 1/(2a'(v))$. Comparing Eq.(6) with Eq.(11) we can see that they are quite different, even after the irrelevant conformal factors $f(u)$ and $g(v)$ are taken into account. Thus, Eq.(6) must be wrong and all the analysis based on it is incorrect.

In addition, the derivation of Eq.(3.8) in [1] is also wrong, because t and r are no longer independent variables once Eq.(3.7) holds. This can also be seen as follows: If Eq.(3.8) were correct, so were Eq.(3.9). Then, from the latter we find $\psi_{,tr} = 2r^{-2}$ and $\psi_{,rt} = 0$, that is, $\psi_{,tr} \neq \psi_{,rt}$, which is clearly wrong. Here Eqs.(3.7), (3.8) and (3.9) are all referred to the ones given in [1].

From Eq.(11), on the other hand, we find

$$\theta_+\theta_- = -\frac{e^{4(\gamma-\psi)}}{4a'(v)b'(u)r^2}, \quad (\alpha = r), \tag{12}$$

which is always negative, as longer as the quantity $(\gamma - \psi)$ is finite. This explains why Gonçalves found no trapped surfaces in the Einstein-Rosen coordinates.

However, there exist cases where $(\gamma - \psi)$ becomes unbounded on a hypersurface and the singularity on this surface is only a coordinate one. Then, to have a geodesically maximal spacetime, extensions beyond this surface are needed. After extending the solution to other regions, trapped surfaces may exist. To show that this possibility indeed exists, let us consider the following solution,

$$\begin{aligned} \gamma &= \frac{2n-1}{4n}\ln\left|\frac{f^4}{t^2-r^2}\right| + \gamma_0, \\ \psi &= q\ln\left|\frac{f}{\sqrt{2}}\right| + \psi_0, \\ \alpha &= r, \quad f = (r-t)^{1/2} + (-t-r)^{1/2}, \end{aligned} \tag{13}$$

where γ_0 and ψ_0 are arbitrary constants, and

$$q \equiv \left(\frac{2n-1}{n}\right)^{1/2}, \tag{14}$$

[1]The functions $a(v)$ and $b(u)$ can be always chosen such that the metric in the (u, v)-coordinates is free of coordinate singularities on hypersurfaces $u = Const.$ or $v = Const.$ This implies that σ is always finite on these hypersurfaces, except for the ones on which the spacetime is singular.

with n being a positive integer, $n \geq 1$. It can be shown that the above solutions satisfy the vacuum Einstein field equations $R_{\mu\nu} = 0$. On the other hand, from Eq.(13) we find

$$X \equiv \xi^{\alpha}_{(3)}\xi^{\beta}_{(3)}g_{\alpha\beta} = r^2 e^{2\psi_0}\left|\frac{f}{\sqrt{2}}\right|^{2q}. \tag{15}$$

Then, it can be shown that the local-flatness condition (2) is satisfied on the symmetry axis $r = 0$, provided that $\gamma_0 = -((2n-1)/n)\ln(2)$. From Eq.(13) we can see that the solutions are valid only in the region $0 \leq r < -t$. The metric coefficient γ is singular on the hypersurface $r = -t$ ($\gamma \to -\infty$, as $r \to -t$) [cf. Fig. 1]. However, this is only a coordinate singularity. To see this, let us choose the functions $a(v)$ and $b(u)$ in Eq.(8) as

$$a(v) = -(-v)^{2n}, \quad b(u) = -(-u)^{2n}, \tag{16}$$

we find that in terms of u and v the metric takes the form

$$ds^2 = -2e^{2\sigma}dudv + e^{2\psi}dz^2 + \alpha^2 e^{-2\psi}d\varphi^2, \tag{17}$$

with

$$\begin{aligned}
\sigma &= -q(1-q)\ln\left|\frac{f}{\sqrt{2}}\right| + \sigma_0,\\
\psi &= q\ln\left|\frac{f}{\sqrt{2}}\right| + \psi_0,\\
\alpha &= r = (-u)^{2n} - (-v)^{2n}, \quad f = (-u)^n + (-v)^n,
\end{aligned} \tag{18}$$

where $\sigma_0 \equiv \frac{1}{2}\ln\left(n^2 2^{(2-n)/n}\right) - \psi_0$. From Eqs.(8) and (16) we can see that the symmetry axis $r = 0$ is mapped to $u = v$, and the region $0 \leq r < -t$ to the one where $0 > u \geq v$, which is referred to as Region II in Fig. 1. The hypersurface $r = -t$ is mapped to $v = 0$, on which the metric coefficients are no longer singularity in the $(u,\ v)$-coordinates. This shows clearly that the singularity on the hypersurface $r = -t$ in the $(t,\ r)$-coordinates is indeed a coordinate one. Region I, where $v > 0,\ u < 0$ and $|u| > v$, is absent in the (t, r)-coordinates, and such represents an extended region. Across the hypersurface $v = 0$ the metric coefficients are analytical, and thus the extension is unique.

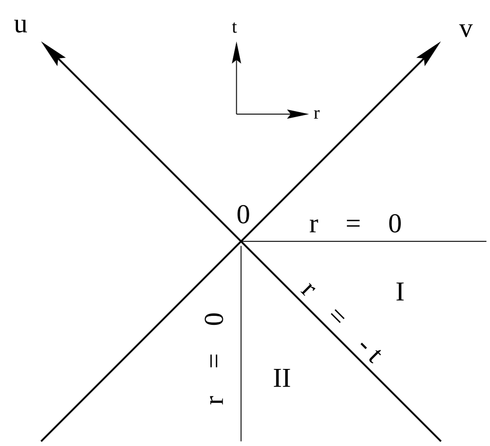

FIG. 1. The spacetime in the (u, v)-plane for the solutions given by Eqs.(17) and (18) in the text. Regions I and II are defined, respectively, as $I = \{x^\alpha : u < 0,\ v > 0,\ |u| > v\}$, and $II = \{x^\alpha : u < 0,\ v < 0,\ |u| > |v|\}$. The spacetime is locally flat on the vertical line $r = 0$, which is the symmetry axis of the spacetime. Region I is absent in the (t, r)-coordinates.

On the other hand, from Eqs.(7) and (18) we find that

$$\begin{aligned}
\theta_+ &= 2ne^{-2\sigma}\frac{(-v)^{2n-1}}{r},\\
\theta_- &= -2ne^{-2\sigma}\frac{(-u)^{2n-1}}{r},
\end{aligned} \tag{19}$$

from which we can see that in Region II, where $u,\ v < 0,\ v > u$, we have $\theta_+ > 0$ and $\theta_- < 0$. Thus, in this region the two-cylinders of constant u and v are untrapped. However, in Region I we have $\theta_+\theta_- > 0$ and the two-cylinders

of constant u and v become trapped. On the hypersurface $v = 0$ we have $\theta_+(u, 0) = 0$ and $\theta_-(u, 0) < 0$. Thus, this hypersurface defines a future apparent horizon. This horizon is degenerate, since now we have $\mathcal{L}_n\theta_+|_{v=0} = 0$. The hypersurface $v = -u$ on which we also have $r = 0$ serves as the up boundary of the spacetime. The nature of the spacetime singularity on this surface depends on the values of n. It can be shown that when n is an odd integer, the spacetime has curvature singularity there, and when n is an even integer, the spacetime is free of curvature singularity, but the local-flatness condition (2) does not hold. That is, in the latter case the spacetime has topological singularity at $u = -v$. The global structure can be seen from the corresponding "Penrose diagram," Fig. 2, where the quotation means that it is only schematic.

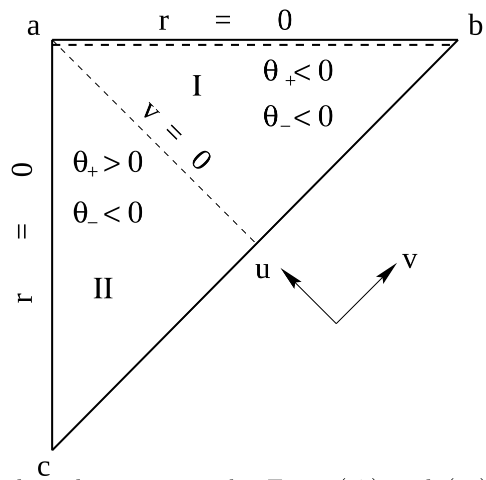

FIG. 2. The "Penrose diagram" for the solutions given by Eqs. (17) and (18) with n being a positive integer. The two-cylinders of constant u and v are trapped in Region I, but not in Region II. The dashed line $v = 0$ represents a future degenerate apparent horizon. When n is an odd integer the spacetime has curvature singularity on the horizontal line $r = 0$, and when n is an even integer it has no curvature singularity but a topological one. The line bc represents the past null infinity $u = -\infty$, and the point a has the coordinates $(u, v) = (0, 0)$.

The above example shows clearly that the Einstein-Rosen coordinates cover only a part of the spacetime, region II, in which we have $\theta_+\theta_- < 0$, that is the two-cylinders of constant t and r are untrapped in this region. However, after the solution is extended to the whole spacetime, trapped surfaces indeed exist in the extended region, I. This is quite similar to the Schwarzschild solution in the spherically symmetric spacetimes.

III. CONCLUSIONS

In this Comment, we have shown that Eq.(6) used by Gonçalves in his first criterion for the existence of trapped surfaces in the cylindrical spacetimes is incorrect [1]. We have also shown that the Einstein-Rosen coordinates used by Gonçalves do not always cover the whole spacetime. As a result, his proof that trapped surfaces don't exist in vacuum is incomplete. In fact, we have presented an example, which is a solution to the Einstein vacuum field equations and satisfies all the conditions imposed by Gonçalves. After extending the solution to the whole spacetime we have shown explicitly that trapped surfaces exist in the extended region.

It should be noted that lately Gonçalves generalized his studies to the case where the cylindrical gravitational waves have two degrees of polarization [9]. Following the arguments given above, one can show that the coordinates used there do not always cover the whole spacetime either, and the analysis for the non-existence of trapped surfaces in the vacuum part of the spacetime is also incomplete.

We would also like to note that the results obtained here do not contradict with the ones obtained by Berger, Chrusciel and Moncrief [10], since in the present case it can be shown that the spacetime is not "asymptotically flat" in the sense defined by them. In addition, our results do not contradict with the ones obtained by Ida either [11]. The reason is simply because Ida defined black hole in terms of outer apparent horizons, while the apparent horizons appearing in the present solutions are degenerate.

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